

# **Big Book of Examples**

Contemporary Mathematics M105

Spring 2020



“This book of examples is dedicated to all of my students that are riding the struggle bus. I’ll save you a seat next to me.”

-Grant

# Contents

<b>1</b>	<b>Exam 1 Topics</b>	<b>2</b>
1.1	Rounding . . . . .	2
1.1.1	Ratios . . . . .	3
1.1.2	Percentages . . . . .	3
1.2	Fractions . . . . .	4
1.2.1	Reducing Fractions . . . . .	4
1.2.2	Multiplying Fractions . . . . .	4
1.2.3	Dividing Fractions . . . . .	5
1.3	Working with Exponents . . . . .	5
1.3.1	Computing Powers . . . . .	6
1.3.2	Undoing Exponents . . . . .	6
1.4	Working with Equations . . . . .	7
1.4.1	Evaluating Equations . . . . .	7
1.4.1.1	Geometric Equations . . . . .	7
1.4.1.2	Equations of the form $\frac{a}{b}x^n \pm c$ . . . . .	9
1.4.2	Solving Equations . . . . .	10
1.4.2.1	Geometric Equations . . . . .	10
1.4.2.2	Equations of the form $\frac{a}{b}x^n \pm c$ . . . . .	11
1.4.2.3	Equations of the form $ax \pm bx = c$ . . . . .	11
1.5	Working with Percentages . . . . .	12
1.5.1	A is P% of B . . . . .	12
1.5.2	A is P% more than B . . . . .	13

1.5.3	A is P% less than B . . . . .	15
1.6	Working with Units . . . . .	16
1.6.1	Using Linear Relationships to obtain Areal and Volumetric Relationships . . . . .	17
1.6.2	The word PER . . . . .	18
1.6.3	Using Unit Multipliers . . . . .	19
1.7	Test 1 Note Sheet . . . . .	22
<b>2</b>	<b>Exam 2 Topics</b>	<b>23</b>
2.1	Key Words . . . . .	23
2.2	Prorating Expenses . . . . .	24
2.3	Simple Interest . . . . .	24
2.4	Compound Interest . . . . .	25
2.4.1	Discretely Compounding Interest . . . . .	25
2.4.2	Continuously Compounding Interest . . . . .	27
2.5	Using Logarithms in order to solve Continuously Compounding Interest	28
2.6	Saving Accounts . . . . .	30
2.7	Amortized Loans . . . . .	32
2.7.1	Mortgages . . . . .	34
2.8	Taxes . . . . .	34
2.8.1	Income Tax . . . . .	35
2.8.2	FICA Tax . . . . .	36
2.8.3	Capital Gains Tax . . . . .	37
2.9	Test 2 Note Sheet . . . . .	40
<b>3</b>	<b>Exam 3 Topics</b>	<b>42</b>
3.1	Counting Methods . . . . .	42

3.1.1	Slot Argument . . . . .	42
3.1.2	Factorial . . . . .	43
3.1.3	${}_nP_r$ . . . . .	43
3.1.4	${}_nC_r$ . . . . .	44
3.2	Probability . . . . .	45
3.2.1	AND Probability . . . . .	47
3.2.1.1	Independent Events . . . . .	47
3.2.1.2	Dependent Events . . . . .	48
3.2.2	OR Probability . . . . .	49
3.2.2.1	Overlapping Events . . . . .	49
3.2.2.2	Non-Overlapping Events . . . . .	50
3.2.3	Expected Outcome . . . . .	50
3.3	Introduction to Statistics . . . . .	51
3.4	Fundamental Statistical Measurements . . . . .	54
3.4.1	Mean, Median, and Mode . . . . .	54
3.4.2	5 Number Summary . . . . .	55
3.4.2.1	Box and Whisker Plot . . . . .	55
3.4.3	Standard Deviation ( $\sigma$ ) . . . . .	56
3.5	Distributions of Data . . . . .	57
3.5.1	Skewed Distributions . . . . .	58
3.5.2	Symmetric Distributions . . . . .	59
3.6	Normal Distributions . . . . .	59
3.6.1	68, 95, 99.7 Rule . . . . .	60
3.6.2	Using Z-chart to Find Percentiles . . . . .	60
3.6.3	Central Limit Theorem . . . . .	61
3.7	Test 3 Note Sheet . . . . .	63

<b>4</b>	<b>Topics after Exam 3</b>	<b>64</b>
4.1	Linear Modeling . . . . .	64
4.2	Exponential Modeling . . . . .	64
4.2.1	Logarithms . . . . .	64
4.2.2	Exponential Growth . . . . .	64
4.2.3	Exponential Decay . . . . .	65

# 1 Exam 1 Topics

## 1.1 Rounding

Solutions need not be a whole number nor a decimal that terminates. For this reason, we must know how to round. First we must recall the name of the decimal places. See the provided table below for some of the decimal place names.

<i>Name</i>	<i>Fraction Representation</i>	<i>Decimal Representation</i>
<i>Integer</i>	$\frac{1}{1}$	1
<i>Tenth</i>	$\frac{1}{10}$	0.1
<i>Hundredth</i>	$\frac{1}{100}$	0.01
<i>Thousandth</i>	$\frac{1}{1000}$	0.001
<i>Ten – Thousandth</i>	$\frac{1}{10,000}$	0.0001

We often refer to a tenth as the first decimal place, a hundredth as the second decimal place and so on. When we round to the  $n^{\text{th}}$  decimal place, we inspect the  $(n + 1)^{\text{th}}$  decimal place and if it is 5 or larger we round the  $n^{\text{th}}$  decimal place up, if not we truncate the decimal.

Let's round some decimals to the stated decimal place.

**Example 1.1.** Round 0.125 to the nearest hundredth.

*Note that the number 2 occupies the hundredth position. Inspecting the digit immediately to the right of 2 we see the number 5. Because 5 is large enough to round up, we round the 2 up to a 3. Hence the solution is 0.13.*

**Example 1.2.** Round 0.196 to the nearest hundredth.

*Note that the number 9 occupies the hundredth position. Inspecting the digit immediately to the right of 9 we see the number 6. Because 6 is large enough to round up, we round the 9 up to a 10. Consequently, the 1 needs to be rounded up to a 2. Hence the solution is 0.20. Note that the solution is 0.20 and not 0.2 because we need to include the hundredth decimal place (because we were told to round to the nearest hundredth).*



### 1.1.1 Ratios

A ratio is a comparison between two numbers by use of division. It is used to tell the relative size of one object with regards to a second object. If the ratio of A to B is 2, then A is twice as large as B. Moreover, if the ratio of A to B is 0.5, then A is half as large as B.

Let's find some ratios and round accordingly.

**Example 1.3.** *Find the ratio of 8 to 69 and round to the nearest hundredth.*

*The ratio of 8 to 69 is  $\frac{8}{69} = 0.1\bar{13}$ . Hence the solution is 0.13*

**Example 1.4.** *Find the ratio of 1 to 169 and round to the nearest hundredth.*

*The ratio of 1 to 169 is  $\frac{1}{169} = 0.0059171\dots$ . Hence the solution is 0.01.*

### 1.1.2 Percentages

A percent is a ratio multiplied by 100 and hence is used to compare two objects. This begs the question, "Why do we use percents if they are simply ratios multiplied by 100?" This is a relevant question. Percents are used in order to avoid awkward statements like "I earned a 0.05 bonus at my job!" and use statements like "I earned a 5% bonus at my job!" instead.

Note that if A is 200% of B, then A is twice as large as B. Also, if A is 50% of B, then A is half the size of B. Moreover, if A is 100% of B, then A is the same size as B.

Let's find some percentages and round accordingly.

**Example 1.5.** *Find the percentage of 123 to 456 and round to the nearest tenth of a percent.*

*Note that the percentage of 123 to 456 is  $\frac{123}{456} \times 100 = 26.9736842\dots$ . Hence the solution is 27.0%.*

## 1.2 Fractions

A fraction contains two pieces: a numerator and denominator. Recall that the numerator is the "upstairs" piece and the denominator is the "downstairs" piece. Hence the fraction  $\frac{a}{b}$  has  $a$  in the numerator position and  $b$  in the denominator position.

Refrigerators refrigerate, terminators terminate, and denominators denominate. To denominate means to measure within a given context. To see this more clearly think of denominations of dollar bills e.g. \$1, \$5, \$10, \$20 etc. Three bills can mean \$3, \$15, \$30, \$60 depending upon the denomination of the bill. In a similar fashion 8 halves is equal to 4 but 8 fourths is equal to 2.

### 1.2.1 Reducing Fractions

In order to make life easier when it comes to multiplying fractions, we need to know how to reduce fractions. Reducing fractions amounts to removing common factors between the numerator and denominator.

Lets see an example of reducing fractions.

**Example 1.6.** Reduce the fraction  $\frac{14}{21}$ .

*Notice,*

$$\frac{14}{21} = \frac{7 \times 2}{7 \times 3} = \frac{2}{3}$$

*which is fully reduced. We can verify that  $\frac{14}{21} = \frac{2}{3}$  by performing the division in our calculator to see that both fractions are equal to  $0.\bar{6}$ .*

### 1.2.2 Multiplying Fractions

Multiplying fractions is easy. Simply multiply top with top and bottom with bottom. Do not confuse yourself and think that a common denominator is needed - that is for adding fractions. Multiplication plays so nicely with fractions because a fraction uses division which is the inverse of multiplication.

Let's see some examples of multiplication with fractions.

**Example 1.7.** Multiply  $\frac{3}{5}$  and  $\frac{7}{11}$  and record your solution as a reduced fraction.

See that,

$$\frac{3}{5} \times \frac{7}{11} = \frac{3 \times 7}{5 \times 11} = \frac{21}{55}$$

which is fully reduced.

**Example 1.8.** Multiply  $\frac{3}{5}$  and  $\frac{50}{9}$  and record your solution as a reduced fraction.

See that

$$\begin{aligned} \frac{3}{5} \times \frac{50}{9} &= \\ \frac{3 \times 1}{5 \times 1} \times \frac{5 \times 10}{3 \times 3} &= \\ \frac{\cancel{3} \times 1}{\cancel{5} \times 1} \times \frac{\cancel{5} \times 10}{\cancel{3} \times 3} &= \\ \frac{1}{1} \times \frac{10}{3} &= \frac{10}{3} \end{aligned}$$

which is fully reduced.

### 1.2.3 Dividing Fractions

Don't divide by a fraction - multiply by the reciprocal.

**Example 1.9.** Divide the fraction  $\frac{3}{5}$  by  $\frac{9}{50}$  and record your solution as a reduced fraction.

See that  $\frac{3}{5}$  divided by  $\frac{9}{50}$  is  $\frac{3}{5} \div \frac{9}{50}$ . At which point recall the trick of multiplying by the reciprocal rather than dividing by a fraction. This gives us,  $\frac{3}{5} \times \frac{50}{9}$ . Hence the solution can be found in the previous example.

## 1.3 Working with Exponents

An exponent is short hand for repeated multiplication. For example,  $a^2$  means  $a \times a$ . See that the exponent tells the reader how many times the multiplication is to be repeated.

### 1.3.1 Computing Powers

**Example 1.10.** Find  $(2^3)^2$ .

See that  $(2^3)^2 = (2 \times 2 \times 2)^2 = (2 \times 2 \times 2) \times (2 \times 2 \times 2) = 2^6$

Notice that we obtained a handy tool here. In general  $(a^b)^c = a^{b \times c}$  This will be useful for undoing exponents.

$$(a^b)^c = a^{b \times c} \tag{1.1}$$

### 1.3.2 Undoing Exponents

We will run into instances where we need to unwind exponents. A simple example is,  $a^2 = 141$ . Notice in order to solve for the variable  $a$  we need to undo the exponent of 2. This is where the square root would be implemented. However, a different but mathematically equivalent approach will be used in order to solve our problem. This approach will leverage the the rule for exponents that we reached in the previous example.

**Example 1.11.** Solve  $a^2 = 141$ .

We wish to undo the exponent of 2. In order to accomplish this we will raise each side of the equation to the  $\frac{1}{2}$  power.

See that,

$$\begin{aligned} a^2 &= 141 \\ (a^2)^{\frac{1}{2}} &= (141)^{\frac{1}{2}} \\ a^{2 \times \frac{1}{2}} &= 141^{\frac{1}{2}} \\ a^1 &= 141^{\frac{1}{2}} \end{aligned}$$

and with the aid of a calculator we see that  $141^{1/2} = 12$ . Hence, we obtain the solution of  $a = 12$ .

Note the general equation used to undo a fixed exponent of  $n$  is given by,

$$a^n = c \iff a = c^{1/n} \tag{1.2}$$

## 1.4 Working with Equations

The key to using equations is to recognize the order of operations. This is usually taught to students with the acronym P.E.M.D.A.S. which stands for parenthesis, exponent, multiplication, division, addition, and subtraction. To compute an equation properly means to evaluate parenthesis followed by exponents, then multiplication, then division, then addition and subtraction.

Let's see some examples.

**Example 1.12.** *Compute  $2(1 + 3)^2 + 2$ .*

$$2(1 + 3)^2 + 2 =$$

$$2(4)^2 + 2 =$$

$$2 \times 16 + 2 =$$

$$32 + 2 = 34$$

**Example 1.13.** *Compute  $2 \times 1 + 3^2 - 2$ .*

$$2 \times 1 + 3^2 - 2 =$$

$$2 \times 1 + 9 - 2 =$$

$$2 + 9 - 2 = 9$$

### 1.4.1 Evaluating Equations

We wish to use equations for the sake of finding relevant information and not just doing algebra for the sake of doing algebra. The process begins with being able to evaluate classic geometric equations.

#### 1.4.1.1 Geometric Equations

The table provided gives equations used to find the area of different two-dimensional geometric shapes.

<i>Shape</i>	<i>Area Formula</i>
<i>Rectangle</i>	$A = L \times W$
<i>Square</i>	$A = s^2$
<i>Circle</i>	$A = \pi \times r^2$
<i>Triangle</i>	$A = \frac{1}{2} \times b \times h$

The table provided gives equations used to find the surface area of different three-dimensional geometric shapes.

<i>Shape</i>	<i>Surface Area Formula</i>
<i>Rectangular Prism</i>	$SA = 2(L \times W) + 2(W \times H) + 2(H \times L)$
<i>Cube</i>	$SA = 6s^2$
<i>Sphere</i>	$SA = 4 \times \pi \times r^2$

The table provided gives equations used to find the volume of different geometric objects.

<i>Shape</i>	<i>Volume Formula</i>
<i>Rectangular Prism</i>	$V = L \times W \times H$
<i>Cube</i>	$V = s^3$
<i>Sphere</i>	$V = \frac{4}{3} \times \pi \times r^3$

Note that an areal formula has two length units being multiplied (because area is a 2-dimensional measurement) whereas volume formulas have three length units being multiplied (because volume is a 3-dimensional measurement).

Let's see some examples of the geometric equations in action.

**Example 1.14.** *A circle has a radius of 2. Compute the area.*

*Note the area equation of a circle for a given radius is,*

$$A = \pi r^2$$

$$A = \pi(2)^2$$

$$A = \pi \times 8$$

$$A = 25.1327412\dots$$

**Example 1.15.** *A sphere has a radius of 2. Compute the surface area.*

*Note the surface area equation of a sphere for a given radius is,*

$$SA = 4\pi r^2$$

$$SA = 4\pi(2)^2$$

$$SA = 4\pi \times 8$$

$$SA = 100.530964\dots$$

**Example 1.16.** *A sphere has a radius of 2. Compute the volume.*

*Note the volume equation of a sphere for a given radius is,*

$$V = \frac{4}{3}\pi r^3$$

$$V = \frac{4}{3}\pi(2)^3$$

$$V = \frac{4}{3}\pi \times 8$$

$$V = 33.5103216\dots$$

#### 1.4.1.2 Equations of the form $\frac{a}{b}x^n \pm c$

Not all situations will follow a geometric formula. For this reason we must know how to work with equations of the form  $\frac{a}{b}x^n \pm c$ . The process of evaluating is the same as it was with the geometric equations. That is, we must follow the rule of P.E.M.D.A.S.

**Example 1.17.** *From the equation  $y = \frac{7}{3}x^3 - 5$  compute  $y$  given  $x = 3$ .*

$$y = \frac{7}{3}x^3 - 5$$

$$y = \frac{7}{3}3^3 - 5$$

$$y = \frac{7}{3}27 - 5$$

$$y = \frac{7}{1}9 - 5$$

$$y = \frac{63}{1} - 5$$

$$y = 63 - 5 = 58$$

## 1.4.2 Solving Equations

### 1.4.2.1 Geometric Equations

Let's use our geometric equations in order to solve for some unknowns.

**Example 1.18.** *A circle has an area of 10. Find the radius.*

$$A = \pi r^2$$

$$10 = \pi r^2$$

$$\frac{10}{\pi} = r^2$$

$$\left(\frac{10}{\pi}\right)^{1/2} = r$$

$$1.78412411\dots = r$$

**Example 1.19.** *A cube with side length 2 has an equal volume to a sphere. What is the radius of the sphere?*

*We begin by computing the volume of the cube.*

$$V = s^3$$

$$V = 2^3 = 8$$

*Recall that the sphere has the same volume as the cube. Hence,*

$$8 = \frac{4}{3}\pi r^3$$

$$8 \times \frac{3}{4\pi} = r^3$$

$$\left(\frac{24}{4\pi}\right)^{1/3} = r$$

$$1.24070098\dots = r$$



### 1.4.2.2 Equations of the form $\frac{a}{b}x^n \pm c$

**Example 1.20.** From the equation  $y = \frac{1}{2}x^3 - 5$ , if  $y = 3$  find the value of  $x$ .

$$3 = \frac{1}{2}x^3 - 5$$

$$8 = \frac{1}{2}x^3$$

$$\frac{2}{1} \times 8 = x^3$$

$$16 = x^3$$

$$(16)^{1/3} = x$$

$$2.519842... = x$$

**Example 1.21.** From the equation  $y = 7x^3 + 5$ , if  $y = 3$  find the value of  $x$ .

$$y = 7x^3 + 5$$

$$3 = 7x^3 + 5$$

$$-2 = 7x^3$$

$$-\frac{2}{7} = x^3$$

$$\left(-\frac{2}{7}\right)^{1/3} = x$$

$$-0.65863375... = x$$

### 1.4.2.3 Equations of the form $ax \pm bx = c$

**Example 1.22.** Solve  $3x + 4x = 14$  for  $x$ .

$$3x + 4x = 14$$

$$7x = 14$$

$$x = \frac{14}{7} = 2$$

**Example 1.23.** Solve  $3x - 4x = 14$  for  $x$ .

$$\begin{aligned}
3x + 4x &= 14 \\
-1x &= 14 \\
x &= \frac{14}{-1} = -14
\end{aligned}$$

## 1.5 Working with Percentages

### 1.5.1 A is P% of B

Some number is 50% of 200, what is the number? It is likely that 100 was an immediate solution that your brain came up with - and it is correct! In order to get our hands on percentages we need to understand what our brain did to come up with the solution of 100. This will allow us to find solutions to more exotic problems such as, “What number is 7.62% of 69?”

The statement “A is P% of B” written mathematically is given by the equation listed below.

$$A = \frac{P}{100} \times B \tag{1.3}$$

Let’s do some examples.

**Example 1.24.** *What number is 68.7% of 31?*

*We use “A is P% of B” in order to obtain,*

$$\begin{aligned}
A &= \frac{P}{100} \times B \\
A &= \frac{68.7}{100} \times 31 \\
A &= 0.687 \times 31 \\
A &= 21.297
\end{aligned}$$

**Example 1.25.** *31 is 68.7% of what number?*

*We use “A is P% of B” in order to obtain,*

$$\begin{aligned}
A &= \frac{P}{100} \times B \\
31 &= \frac{68.7}{100} \times B
\end{aligned}$$

$$\begin{aligned}
31 &= 0.687 \times B \\
\frac{31}{0.687} &= B \\
45.1237263\dots &= B
\end{aligned}$$

**Example 1.26.** *13 is what percent of 31?*

We use “A is P% of B” in order to obtain,

$$\begin{aligned}
A &= \frac{P}{100} \times B \\
13 &= \frac{P}{100} \times 31 \\
\frac{13}{31} &= \frac{P}{100} \\
0.41935483\dots &= \frac{P}{100} \\
0.41935483\dots \times 100 &= P \\
41.935483\dots &= P
\end{aligned}$$

### 1.5.2 A is P% more than B

It is a natural occurrence for quantities to grow in relation to an original reference value. For instance a 5% raise in wages, a 30% increase in retail price from the wholesale price, or a 2% growth in population. These percentages are all remarking on a growth on top of an original quantity. Solving equations such as “A is P% more than B” starts by recognizing that a quantity is growing and fitting it into the form “A is P% more than B” so that the numbers reflect the situation at hand.

The statement “A is P% more than B” written mathematically is given by the equations listed below.

$$A = B + \frac{P}{100} \times B \tag{1.4}$$

$$A = B \left( 1 + \frac{P}{100} \right) \tag{1.5}$$

Let’s do some examples.

**Example 1.27.** *What number is 68.7% more than 31?*

*We use “A is P% more than B” in order to obtain,*

$$\begin{aligned}A &= B + \frac{P}{100} \times B \\A &= 31 + \frac{68.7}{100} \times 31 \\A &= 31 + 0.687 \times 31 \\A &= 52.297\end{aligned}$$

**Example 1.28.** *31 is 68.7% more than what number?*

*We use “A is P% more than B” in order to obtain,*

$$\begin{aligned}A &= B + \frac{P}{100} \times B \\31 &= B + \frac{68.7}{100} \times B \\31 &= B + 0.687 \times B \\31 &= 1.687 \times B \\ \frac{31}{1.687} &= B \\18.3758150\dots &= B\end{aligned}$$

**Example 1.29.** *31 is what percent more than 13?*

*We use “A is P% more than B” in order to obtain,*

$$\begin{aligned}A &= B + \frac{P}{100} \times B \\31 &= 13 + \frac{P}{100} \times 13 \\18 &= \frac{P}{100} \times 13 \\ \frac{18}{13} &= \frac{P}{100} \\ \frac{18}{13} \times 100 &= P \\138.461538\dots &= P\end{aligned}$$

### 1.5.3 A is P% less than B

It is just as natural for quantities to decrease with respect to an original reference value. For example, a 10% discount at the store, a 2% population decrease, or a 26% tax on wages. All of these percentages are referencing an original amount for a net decrease of the original quantity.

The statement “A is P% less than B” written mathematically is given by the equations listed below.

$$A = B - \frac{P}{100} \times B \quad (1.6)$$

$$A = B \left( 1 - \frac{P}{100} \right) \quad (1.7)$$

Let’s do some examples.

**Example 1.30.** *What number is 68.7% less than 31?*

*We use “A is P% less than B” in order to obtain,*

$$\begin{aligned} A &= B - \frac{P}{100} \times B \\ A &= 31 - \frac{68.7}{100} \times 31 \\ A &= 31 - 0.687 \times 31 \\ A &= 9.703 \end{aligned}$$

**Example 1.31.** *31 is 68.7% less than what number?*

*We use “A is P% less than B” in order to obtain,*

$$\begin{aligned} A &= B - \frac{P}{100} \times B \\ 31 &= B - \frac{68.7}{100} \times B \\ 31 &= B - 0.687 \times B \\ 31 &= (1 - 0.687) \times B \\ 31 &= 0.313 \times B \end{aligned}$$

$$\frac{31}{0.313} = B$$

$$99.0415335... = B$$

**Example 1.32.** *13 is what percent less than 31?*

We use “A is P% less than B” in order to obtain,

$$A = B - \frac{P}{100} \times B$$

$$13 = 31 - \frac{P}{100} \times 31$$

$$-18 = -\frac{P}{100} \times 31$$

$$-\frac{18}{31} = -\frac{P}{100}$$

$$-\frac{18}{31} \times 100 = -P$$

$$58.0645161... = P$$

## 1.6 Working with Units

We wish to attach real world meaning to our numbers. To do this we must work with units. Interestingly enough we can treat units much like we do numbers in the sense that we may add, subtract, multiply, and divide units. To see this more clearly, inspect the following example.

**Example 1.33.** *Find the volume of a sphere with a radius of 2 inches.*

$$V = \frac{4}{3}\pi r^3$$

$$V = \frac{4}{3}\pi \times (2 \text{ in.})^3$$

$$V = \frac{4}{3}\pi \times 8 \text{ in.}^3$$

$$V = 33.5103216... \text{ in.}^3$$

Notice in the example above where we cubed our radius. The radius was given in inches (a one-dimensional length measurement) and upon cubing we obtained cubic inches (a three-dimensional volumetric measurement).

We start our investigation of unit analysis by using linear relationships such as 12 inches equals 1 foot in order to obtain areal and volumetric relationships such as 144 square inches equals 1 square foot and 1728 cubic inches equals 1 cubic foot, respectively.

### 1.6.1 Using Linear Relationships to obtain Areal and Volumetric Relationships

Note the following linear relationships provided below.

Table 1.1: Linear Relationships

$$12 \text{ inches} = 1 \text{ foot}$$

$$3 \text{ feet} = 1 \text{ yard}$$

$$5280 \text{ feet} = 1 \text{ mile}$$

$$1 \text{ mile} = 1609 \text{ meters}$$

Let's see how these linear relationships allow us to find relationships of area and volume.

**Example 1.34.** *How many square feet are in a square yard?*

$$3 \text{ feet} = 1 \text{ yard}$$

$$(3 \text{ feet})^2 = (1 \text{ yard})^2$$

$$3^2 \text{ feet}^2 = 1^2 \text{ yard}^2$$

$$9 \text{ feet}^2 = 1 \text{ yard}^2$$

**Example 1.35.** *How many cubic feet are in a cubic yard?*

$$3 \text{ feet} = 1 \text{ yard}$$

$$(3 \text{ feet})^3 = (1 \text{ yard})^3$$

$$3^3 \text{ feet}^3 = 1^3 \text{ yard}^3$$

$$27 \text{ feet}^3 = 1 \text{ yard}^3$$

## 1.6.2 The word PER

Another fundamental idea to unit analysis is the recognition of the word “PER” which means to divide. You have likely heard of the following measurements of miles per gallon, people per household, or mass per volume which measure engine efficiency, household size, and density respectively.

Let’s use the word “PER” in the following examples.

**Example 1.36.** *A car went 241 miles and used 11.2 gallons of gas. Compute the car’s fuel efficiency in miles per gallon.*

*See that we want a measure of  $\frac{\text{miles}}{\text{gallons}}$ . So we have,*

$$\frac{241 \text{ miles}}{11.2 \text{ gallons}} = 21.517 \frac{\text{miles}}{\text{gallon}}$$

**Example 1.37.** *A country has 1.3 million people and 102,456 square miles of land. Calculate the population density in people per square mile.*

*See that we are looking for a measurement of  $\frac{\text{people}}{\text{square mile}}$ . So we have,*

$$\frac{1,300,000 \text{ people}}{102,456 \text{ square miles}} = 12.688 \frac{\text{people}}{\text{square mile}} \approx 13 \frac{\text{people}}{\text{square mile}}$$

*population density*

**Example 1.38.** *A cube has a side length of 2 inches and a mass of 16 grams. Find the density of the cube in grams per cubic inch.*

*See that we are looking for density which is defined as  $\frac{\text{mass}}{\text{volume}}$ , where the volume is measured in cubic inches and the mass is measured in grams. So we have,*

$$V = s^3 = (2 \text{ in.})^3 = 8 \text{ in.}^3$$

*Now we will use the fact that density is  $\frac{\text{mass}}{\text{volume}}$ .*

$$\frac{\text{Mass}}{\text{Volume}} = \frac{16 \text{ grams}}{8 \text{ in.}^3} = 2 \frac{\text{grams}}{\text{in.}^3}$$



### 1.6.3 Using Unit Multipliers

We often need to adjust units that are provided into units that we want. For example, if someone ran a 100 meter race in 12 seconds we could obtain his/her speed in meters per second quite easily. However, if we want to know the speed in miles per hour then we need to implement unit multipliers.

Relationships of time are provided in Table 1.2. These will be needed, as well as the relationships listed in Table 1.1, for unit multipliers.

Table 1.2: Time Relationships

$$60 \text{ seconds} = 1 \text{ minute}$$

$$60 \text{ minutes} = 1 \text{ hour}$$

$$24 \text{ hours} = 1 \text{ day}$$

$$365 \text{ days} = 1 \text{ year}$$

Provided in Table 1.3 are common formulas that describe physical relationships which we will be using during our study of unit analysis.

Table 1.3: Physical Relationships

$$Time = \frac{Distance}{Speed}$$

$$Distance = Speed \times Time$$

$$Speed = \frac{Distance}{Time}$$

$$Density = \frac{Mass}{Volume}$$

$$Mass = Density \times Volume$$

$$Volume = \frac{Mass}{Density}$$

Let's see some examples of unit multipliers at work.

**Example 1.39.** *Usain Bolt ran the 100 meter dash in 9.58 seconds. Calculate his speed in miles per hour.*

*Note that speed is defined as distance per time. We will insert the given values into the description of speed.*

$$Speed = \frac{Distance}{Time} = \frac{100 \text{ meters}}{9.58 \text{ seconds}}$$

*We now implement the linear and time relationships introduced in Table 1.1 and 1.2 in order to convert our  $\frac{\text{meters}}{\text{second}}$  into  $\frac{\text{miles}}{\text{hour}}$  by use of unit multipliers.*

$$\begin{aligned} \frac{100 \text{ meters}}{9.58 \text{ seconds}} &\times \frac{60 \text{ seconds}}{1 \text{ minute}} \times \frac{60 \text{ minutes}}{1 \text{ hour}} \times \frac{1 \text{ mile}}{1609 \text{ meters}} = \\ &\frac{100 \times 60 \times 60 \times 1 \text{ miles}}{9.58 \times 1 \times 1 \times 1609 \text{ hour}} = \\ &\frac{360,000 \text{ miles}}{15,414.22 \text{ hour}} \approx 23.36 \frac{\text{miles}}{\text{hour}} \end{aligned}$$

**Example 1.40.** *A bullet travels 2,300 feet per second. Find the time it takes, in seconds, for the bullet to travel 100 yards.*

*Note that time is defined to be a distance divided by a speed. Hence we have,*

$$Time = \frac{Distance}{Speed} = \frac{100 \text{ yards}}{\left(\frac{2,300 \text{ feet}}{1 \text{ second}}\right)}$$

Note that we do not divide by a fraction, we multiply by the reciprocal. Hence we have,

$$\frac{100 \text{ yards}}{\left(\frac{2,300 \text{ feet}}{1 \text{ second}}\right)} = \frac{100 \text{ yards}}{1} \times \frac{1 \text{ second}}{2,300 \text{ feet}}$$

Now we introduce our unit multiplies in order to get units of miles per hour.

$$\begin{aligned} \frac{100 \text{ yards}}{1} \times \frac{1 \text{ second}}{2,300 \text{ feet}} \times \frac{3 \text{ feet}}{1 \text{ yard}} &= \\ \frac{100 \times 3 \text{ seconds}}{2,300} &\approx 0.1304 \text{seconds} \end{aligned}$$

## 1.7 Test 1 Note Sheet

### Test 1 Note Sheet

---

$$time = \frac{distance}{speed}$$

$$distance = speed \times time$$

$$speed = \frac{distance}{time}$$

$$density = \frac{mass}{volume}$$

$$mass = density \times volume$$

$$volume = \frac{mass}{density}$$

$$1 \text{ mile} = 1609 \text{ meters}$$

$$1 \text{ mile} = 5280 \text{ feet}$$

$$12 \text{ inches} = 1 \text{ foot}$$

$$3 \text{ feet} = 1 \text{ yard}$$

$$1 \text{ yard}^3 = 27 \text{ feet}^3$$

$$60 \text{ seconds} = 1 \text{ minute}$$

$$60 \text{ minutes} = 1 \text{ hour}$$

$$24 \text{ hours} = 1 \text{ day}$$

$$365 \text{ days} = 1 \text{ year}$$

## 2 Exam 2 Topics

### 2.1 Key Words

When working with finances it is convenient to think of segments of a year. Below are listed terminologies used to refer to segments of a year.

- (1) Annually is a full year. If an occurrence happens annually then it happens once per year.
- (2) Semiannually is six months. If an occurrence happens semiannually then it happens twice per year.
- (3) Quarterly is every three months. If an occurrence happens quarterly then it happens four times per year.
- (4) Monthly is every month. If an occurrence happens monthly then it happens twelve times per year.
- (5) Daily is every day. If an occurrence happens daily then it happen 365 times per year.

There are a few other key words that we should familiarize ourselves with in order to work with finances. The list below defines other key words in finance.

- (1) Principal: Within the context of saving money, the principal is the amount invested. Within the context of borrowing money, the principal is the amount borrowed.
- (2) Interest: Within the context of saving money, interest can be thought of as the value of saving. Within the context of borrowing money, interest can be thought of as the cost of borrowing money.

## 2.2 Prorating Expenses

To prorate means to allocate or distribute. It is common to get paid an annual salary but it is helpful to know the monthly salary; to do this we prorate the annual salary to a monthly salary. When prorating we commonly use the following relationships:

Table 2.1: Prorating Relationships

$$\begin{aligned} 52 \text{ weeks} &= 1 \text{ year} \\ 4 \text{ weeks} &= 1 \text{ month} \\ 12 \text{ months} &= 1 \text{ year} \end{aligned}$$

Let's see an example.

**Example 2.1.** Use the given table in order to compute the net monthly cash flow.

Income	Expenses
Part-time job: 100 weekly	Rent: 500 monthly
College fund: 200 monthly	Groceries: 70 weekly
Scholarship: 6600 yearly	Tuition and fees: 3000 semiannually
	Incidentals: 60 weekly

We solve for the net monthly cash flow by the use of Table 2.1 and the definitions provided at the beginning of this chapter.

$$\begin{aligned} \text{Net Monthly Cash Flow} &= \text{Monthly Income} - \text{Monthly Expenses} \\ &= \left( 100 \times 4 + 200 + \frac{6600}{12} \right) - \left( 500 + 70 \times 4 + \frac{3000}{6} + 60 \times 4 \right) \end{aligned}$$

To avoid rounding errors we type the above equation into our calculator and compute in one step.

$$= -370.00$$

## 2.3 Simple Interest

Simple interest is calculated from the original principal only. The amount saved in an account that pays simple interest is given by the formula,

$$A = P + P \times APR \times Y \quad (2.1)$$

where  $A$  is the amount saved,  $P$  is the principal invested,  $APR$  is the annualized percentage rate, and  $Y$  is the number of years that the principal has been in the account.

Let's do an example.

**Example 2.2.** *An investment of \$100 is put into an account that pays simple interest with an APR of 5.5%. Compute the amount in the account at years 1, 2, and 3.*

*From the equation  $A = P + P \times APR \times Y$  we have,*

$$A = 100 + 100 \times 0.055 \times 1 = 105.50$$

$$A = 100 + 100 \times 0.055 \times 2 = 111.00$$

$$A = 100 + 100 \times 0.055 \times 3 = 116.50$$

*Note that the APR is a percent and hence we must divide by 100 in order to obtain the decimal notation which is utilized in the calculations. Also, note that we carry our solutions to the hundredth decimal place (which represents the number of pennies) because we are working with money.*

## 2.4 Compound Interest

### 2.4.1 Discretely Compounding Interest

Simple interest is generated from the original principal only. Compound interest uses the previous cycle's amount in order to generate the interest for the next compounding cycle. The amount in an account that pays compound interest is given by the equation listed below.

$$A = P \left( 1 + \frac{APR}{n} \right)^{nY} \quad (2.2)$$

Note that  $A$  is the amount saved,  $P$  is the principal invested,  $APR$  is the annualized percentage rate, and  $Y$  is the number of years that the principal has been in

the account, and  $n$  is the number of times the interest compounds per year (see the definitions at the beginning of the chapter for values of  $n$ ).

Let's see some examples.

**Example 2.3.** *A person invests \$121 in an account that pays an APR of 5.5% which compounds annually. How much is in the account after 11 years?*

From the equation  $A = P \left( 1 + \frac{APR}{n} \right)^{nY}$  we have,

$$A = 121 \left( 1 + \frac{0.055}{1} \right)^{1 \times 11}$$
$$A = \$218.05$$

**Example 2.4.** *A person invests \$121 in an account that pays an APR of 5.5% which compounds semiannually. How much is in the account after 11 years?*

From the equation  $A = P \left( 1 + \frac{APR}{n} \right)^{nY}$  we have,

$$A = 121 \left( 1 + \frac{0.055}{2} \right)^{2 \times 11}$$
$$A = \$219.78$$

**Example 2.5.** *A person invests \$121 in an account that pays an APR of 5.5% which compounds quarterly. How much is in the account after 11 years?*

From the equation  $A = P \left( 1 + \frac{APR}{n} \right)^{nY}$  we have,

$$A = 121 \left( 1 + \frac{0.055}{4} \right)^{4 \times 11}$$
$$A = \$220.67$$

**Example 2.6.** *A person invests \$121 in an account that pays an APR of 5.5% which compounds monthly. How much is in the account after 11 years?*

From the equation  $A = P \left( 1 + \frac{APR}{n} \right)^{nY}$  we have,



$$A = 121 \left( 1 + \frac{0.055}{12} \right)^{12 \times 11}$$

$$A = \$221.28$$

**Example 2.7.** *A person invests \$121 in an account that pays an APR of 5.5% which compounds daily. How much is in the account after 11 years?*

From the equation  $A = P \left( 1 + \frac{APR}{n} \right)^{nY}$  we have,

$$A = 121 \left( 1 + \frac{0.055}{365} \right)^{365 \times 11}$$

$$A = \$221.57$$

Notice, from the above examples, that the more frequently the interest compounds the larger the return in the investment.

## 2.4.2 Continuously Compounding Interest

Continuously compounding interest is when the interest compounds non-stop. The amount in an account that pays continuously compounding interest is given by the equation listed below.

$$A = Pe^{APR \times Y} \tag{2.3}$$

Note that this equation uses the same ingredients as the compounding interest equation with one exception; in the continuously compounding interest equation we have the number  $e$ . Every calculator should have the number  $e$  on it, or more commonly an “ $e \wedge ( \quad )$ ” button. The number  $e$  is a special number much like  $\pi$ , where  $e = 2.71828182\dots$

Let’s see an example.

**Example 2.8.** *A person invests \$121 in an account that pays an APR of 5.5% which compound continuously. How much is in the account after 11 years?*

From the equation  $A = Pe^{APR \times Y}$  we have,

$$A = 121e^{0.055 \times 11}$$

$$A = \$221.58$$

Notice, yet again, because the interest compounds most frequently we have the largest return for continuously compounding interest.

## 2.5 Using Logarithms in order to solve Continuously Compounding Interest

We now take a peek at logarithms which will be studied more completely later. For now, we will acquaint ourselves with logarithms so they are not so intimidating when we study exponential modeling. Let it be known that a logarithm is used to solve for a variable that is located in the exponent position. We will look at logarithms being utilized within familiar equations in order to establish a foundation for our discussion.

**Example 2.9.** *Three raised to what power is nine?*

$$3^? = 9$$

$$\log_3(3^?) = \log_3(9)$$

$$? = \log_3(9)$$

$$? = 2$$

**Example 2.10.** *Five raised to what power is one-hundred-twenty-five.*

$$2 \times 5^? = 250$$

$$5^? = \frac{250}{2}$$

$$5^? = 125$$

$$\log_5(5^?) = \log_5(125)$$

$$? = \log_5(125)$$

$$? = 3$$

Notice that the logarithm is used to solve for a variable in the exponent position. Also notice the base of the logarithm is the same as the base which contains the

variable and it is introduced directly on the base (there is no multiplicative factor on the base when the logarithm comes into play). We commonly appeal to our calculator when we evaluate something like  $\log_3(9)$  or  $\log_5(125)$ .

We now bring our attention back to the context of finance. It is natural to ask, “If \$100 is invested into an account that compounds continuously with an APR of 4.2%, how long will it take to double the investment?” This is precisely the type of question we wish to answer by the utilization of logarithms.

Let’s answer this question in the following example with use of a logarithm with base  $e$ . A logarithm with base  $e$  is so common that it gets its own notation which is noted below.

$$\log_e(a) = \ln(a)$$

**Example 2.11.** *If \$100 is invested into an account that compounds continuously with an APR of 4.2%, how long will it take to double the investment?*

$$A = Pe^{APR \times Y}$$

$$200 = 100e^{0.042 \times Y}$$

$$\frac{200}{100} = e^{0.042 \times Y}$$

$$\ln\left(\frac{200}{100}\right) = \ln(e^{0.042 \times Y})$$

$$\ln\left(\frac{200}{100}\right) = 0.042 \times Y$$

$$\frac{\ln\left(\frac{200}{100}\right)}{0.042} = Y$$

$$16.5 = Y$$

**Example 2.12.** *If \$100 is invested into an account that compounds continuously with an APR of 4.2%, how long will it take to gain \$120?*

$$A = Pe^{APR \times Y}$$

$$\begin{aligned}
220 &= 100e^{0.042 \times Y} \\
\frac{220}{100} &= e^{0.042 \times Y} \\
\ln\left(\frac{220}{100}\right) &= \ln(e^{0.042 \times Y}) \\
\ln\left(\frac{220}{100}\right) &= 0.042 \times Y \\
\frac{\ln\left(\frac{220}{100}\right)}{0.042} &= Y \\
18.77 &= Y
\end{aligned}$$

We have obtained another useful generalization. The number of years  $Y$  to attain the amount  $A$  for an investment  $P$  with a given  $APR$  is given by the following equation.

$$Y = \frac{\ln\left(\frac{A}{P}\right)}{APR} \quad (2.4)$$

## 2.6 Saving Accounts

We now investigate savings accounts. The equation used to compute the amount in a savings account is given by,

$$A = PMT \times \frac{\left[ \left(1 + \frac{APR}{n}\right)^{(nY)} - 1 \right]}{\left(\frac{APR}{n}\right)} \quad (2.5)$$

where  $A$  is the amount saved in the account (including interest),  $PMT$  is the regular deposit made into the savings account,  $APR$  is the annualized percentage rate,  $n$  is

the number of times per year the deposit is made, and  $Y$  is the number of years the deposits are made.

Let's see some examples.

**Example 2.13.** *A person makes regular monthly deposits of \$70 dollars for ten years into an account with an APR of 2.3%. How much has the person made from the use of this savings account?*

*Recognize that we must find the total amount in the savings account after ten years and remove the total amount in deposits over the ten years.*

$$A = PMT \times \frac{\left[ \left( 1 + \frac{APR}{n} \right)^{(nY)} - 1 \right]}{\left( \frac{APR}{n} \right)}$$

$$A = 70 \times \frac{\left[ \left( 1 + \frac{0.023}{12} \right)^{(12 \times 10)} - 1 \right]}{\left( \frac{0.023}{12} \right)}$$

$$A = 9,434.40$$

*Now we find the total amount deposited. See that 12 deposits were made per year for 10 years in the amount of 70 dollars.*

$$A = 12 \times 10 \times 70 = 8,400$$

*So this person has made,*

$$9,434.40 - 8,400 = 1,034.40$$

Now we see a natural example of where savings accounts may be useful.

**Example 2.14.** *A person is 30 years old wants to retire at 65 and have \$100,000 to live off of every year forever. What monthly deposit is required to achieve this goal if a constant APR of 2.5% is assumed?*

*We first find the amount that we must save in order to have \$100,000 of interest generated every year.*

$$\begin{aligned}
& (\text{Amount Saved}) \times APR = \text{Interest} \\
& (\text{Amount Saved}) \times 0.025 = 100,000 \\
& \text{Amount Saved} = \frac{100,000}{0.025} = 4,000,000
\end{aligned}$$

Hence, 4 million dollars must be saved. We now compute the monthly deposit required to save 4 million dollars in 35 years.

From our savings account equation,

$$A = PMT \times \frac{\left[ \left( 1 + \frac{APR}{n} \right)^{(nY)} - 1 \right]}{\left( \frac{APR}{n} \right)}$$

we have the payment ( $PMT$ ) equalling,

$$A \times \frac{\left( \frac{APR}{n} \right)}{\left[ \left( 1 + \frac{APR}{n} \right)^{(nY)} - 1 \right]} = PMT$$

with the given information we obtain,

$$PMT = 4,000,000 \times \frac{\left( \frac{0.025}{12} \right)}{\left[ \left( 1 + \frac{0.025}{12} \right)^{(12 \times 35)} - 1 \right]}$$

$$PMT = 5,966.48$$

## 2.7 Amortized Loans

We now bring our attention to amortized loans. In banking and finance, an amortizing loan is a loan where the principal of the loan is paid down over the life of the loan according to an amortization schedule, typically through equal payments.

Auto loans, home loans (mortgages), student loans are some typical examples of amortized loans. Text books may or may not make mention of this, but it is important that when a borrower has an amortized loan that the borrower makes more than the minimum payment and request that the extra payment is put on the principal of the loan. A small extra payment can save the borrow a substantial amount of money.

The following equation is used to compute the payment for an amortized loan.

$$PMT = P \times \frac{\left(\frac{APR}{n}\right)}{\left[1 - \left(1 + \frac{APR}{n}\right)^{(-nY)}\right]} \quad (2.6)$$

For now, we focus on computing payments for an amortized loan.

**Example 2.15.** *A student loan has a principal of \$25,000 with an APR of 5.65%. The loan is to be paid back with monthly payments for 15 years. Compute the monthly payment. Also compute the total paid. From the total paid, what percent went toward paying back the principal and what percent went towards paying interest?*

*From the equation,*

$$PMT = P \times \frac{\left(\frac{APR}{n}\right)}{\left[1 - \left(1 + \frac{APR}{n}\right)^{(-nY)}\right]}$$

*we have,*

$$PMT = 25,000 \times \frac{\left(\frac{0.0565}{12}\right)}{\left[1 - \left(1 + \frac{0.0565}{12}\right)^{(-12 \times 15)}\right]}$$

$$PMT = 206.27$$

*We now compute the total paid over the life of the loan.*

$$Total = 206.27 \times 12 \times 15 = 37,128.60$$

We now find the percent principal. See that 25,000 dollars went towards the principal (the amount borrowed).

$$\frac{25000}{37128.60} = 0.67333 \implies 67.33\%$$

Thus the percent interest is,

$$100 - 67.33 = 32.67\%$$

### 2.7.1 Mortgages

We wish to explore a common type of amortized loan, mortgages. A mortgage is an amortized loan specifically for buying a 4-walled dwelling (a home). We wish to gain some important terminologies within the process of buying a home. We begin by introducing some important key words.

- (1) List Price: The price of the home.
- (2) Points: One point is one percent of the list price of the home.
- (3) Down Payment: Money that must be paid upfront. This payment reduces the amount that is financed for the home purchase.
- (4) Principal Borrowed: List price minus the down payment.
- (5) Closing Costs: Money that is paid upfront for the title company, home appraisal, home inspection, loan underwriter, amongst other entities.
- (6) Mortgage Insurance: Required insurance for the amortized loan if the down payment is less than 20 points.
- (7) Home Insurance: Required insurance to protect the finance companies investment (your home).

These definitions will be useful when exploring mortgages.

## 2.8 Taxes

We will explore a variety of taxes: income, FICA, and capital gains.



## 2.8.1 Income Tax

Income tax is collected from one's taxable income. We Use the provided flow chart to find one's taxable income.

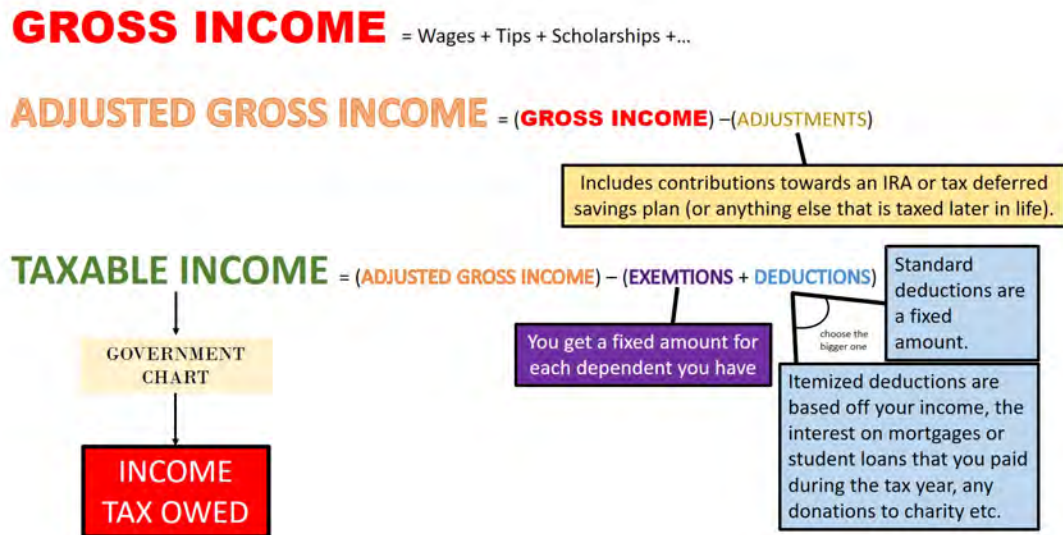


Figure 2.1: A flow chart of tax terminologies

Notice in finding taxable income we want to maximize our deductions. A standard deduction is worth 6,100 dollars. So if the itemized deductions are less than 6,100 we want to chose the standard deduction. Also, notice that we get an exemption for each dependent (yourself is considered as a dependent).

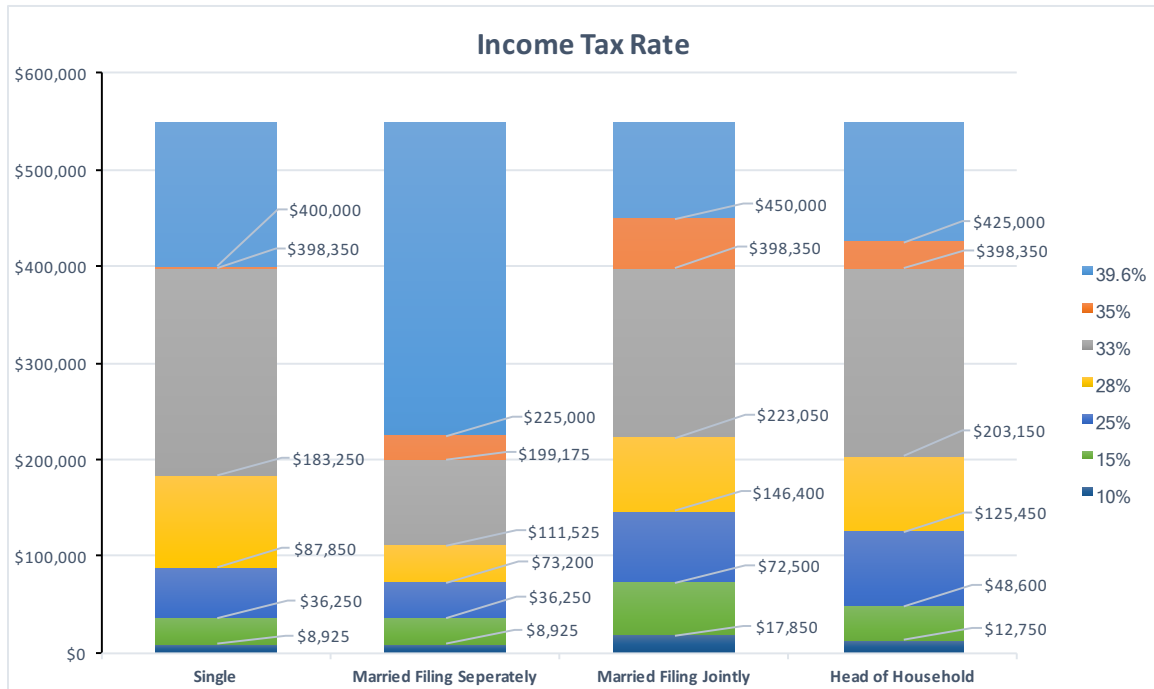


Figure 2.2: Income Tax Brackets

### 2.8.2 FICA Tax

FICA (Federal Insurance Contributions Act) Tax funds social security and medicare. FICA taxes are paid by both the employee and employer and hence if you are self-employed, then you pay FICA taxes at twice the rate. Also, FICA taxes are computed from the gross salary and no exemptions or deductions may be counted towards FICA tax. A FICA tax bracket is provided in Figure 2.3.

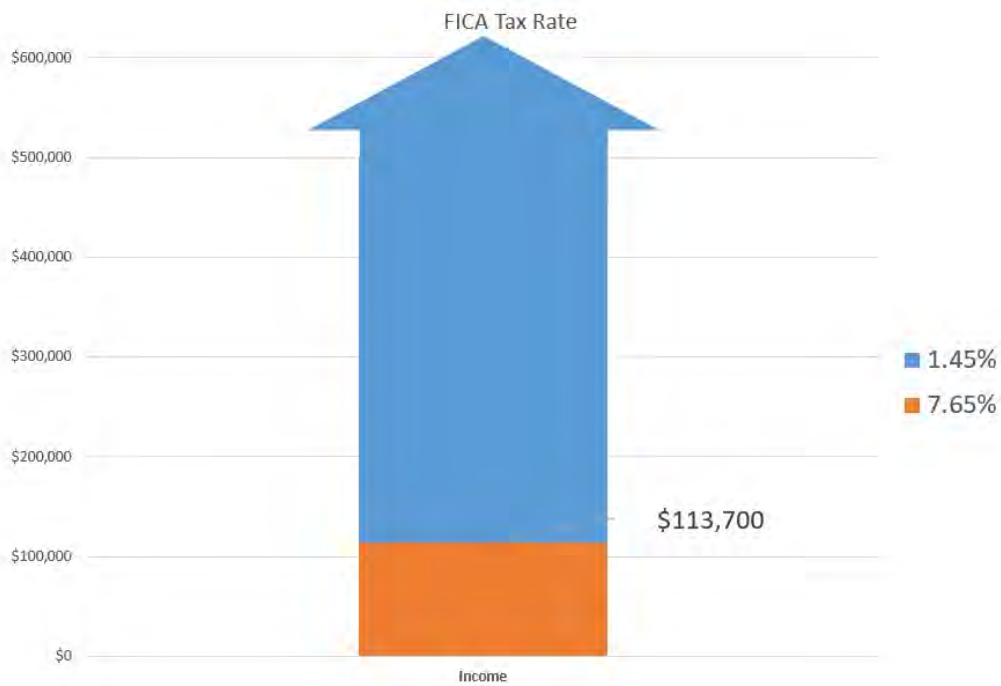


Figure 2.3: FICA Tax Bracket

### 2.8.3 Capital Gains Tax

Capital gains tax is a tax on any good that is sold for a profit. If the item was owned for less than a year before the selling, it is considered short term capital gains and is subject to tax at the same rate as income tax. If the item was owned for more than a year before the selling, it is considered long term capital gains and is subject to the tax at the rate shown in Figure 2.4.

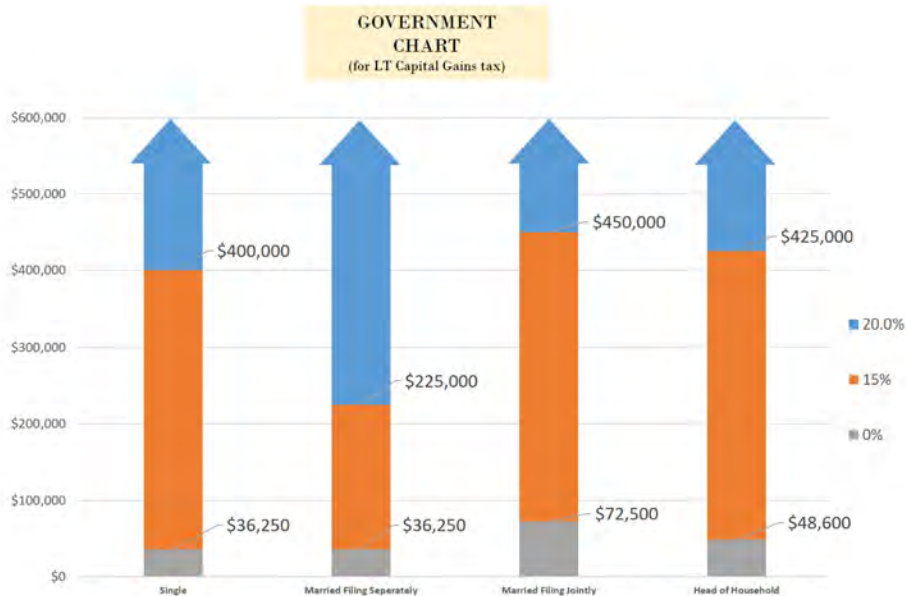


Figure 2.4: Long Term Capital Gains Tax Brackets

**Example 2.16.** A man earned \$48,700 in wages and tips. He had no other income and made no contributions to retirement plans. He is single and takes the standard deduction. Calculate the FICA taxes and income taxes to obtain a total tax bill. Then find the overall tax rate on the gross income, including both FICA and income taxes.

We first find the FICA tax owed by using Figure 2.3. Because the gross income is less than \$113,700 we find 7.65% of the gross salary.

$$48,700 \times 0.0765 = 3,725.55$$

We now calculate the income taxes by use of Figure 2.2. To do so we must first find the taxable income by use of Figure 2.1.

$$\text{Gross Income} = 48,700$$

$$\text{Adjusted Gross Income} = \text{Gross Income} - \text{Adjustments}$$

$$\text{Adjusted Gross Income} = 48,700 - 0$$

$$\text{Taxable Income} = 48,700 - \text{Exemption} - \text{Deduction}$$

$$\text{Taxable Income} = 48,700 - 3,900 - 6,100 = 38,700$$

So the income tax owed is,

$$\text{Income Tax Owed} = 0.25(38,700 - 36,250) + 0.15(36,250 - 8,925) + 0.10(8,925)$$

$$\text{Income Tax Owed} = 5,603.75$$

We now calculate his total tax bill.

$$\text{Total Taxes Owed} = \text{FICA Tax} + \text{Income Tax}$$

$$\text{Total Taxes Owed} = 3,725.55 + 5,603.75 = 9,329.30$$

Hence his overall tax rate is,

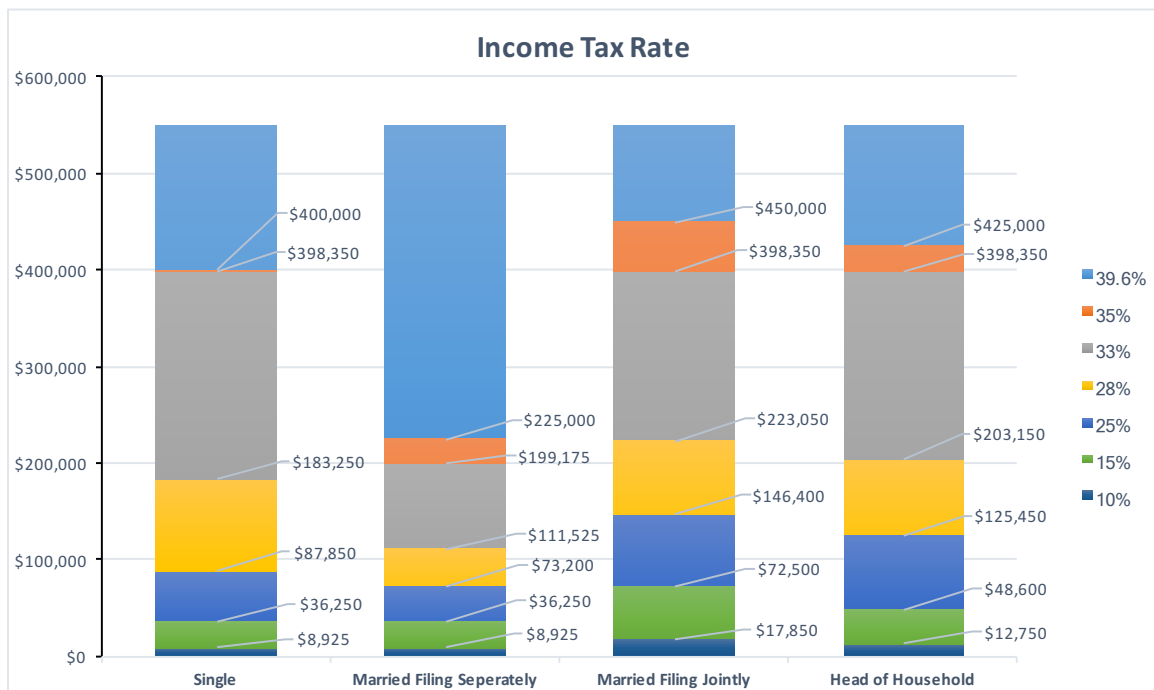
$$\frac{\text{Total Taxes Owed}}{\text{Gross Salary}} = \frac{9,329.30}{48,700} = 0.19156 \implies 19.16\%$$

## 2.9 Test 2 Note Sheet

### Test 2 Note Sheet

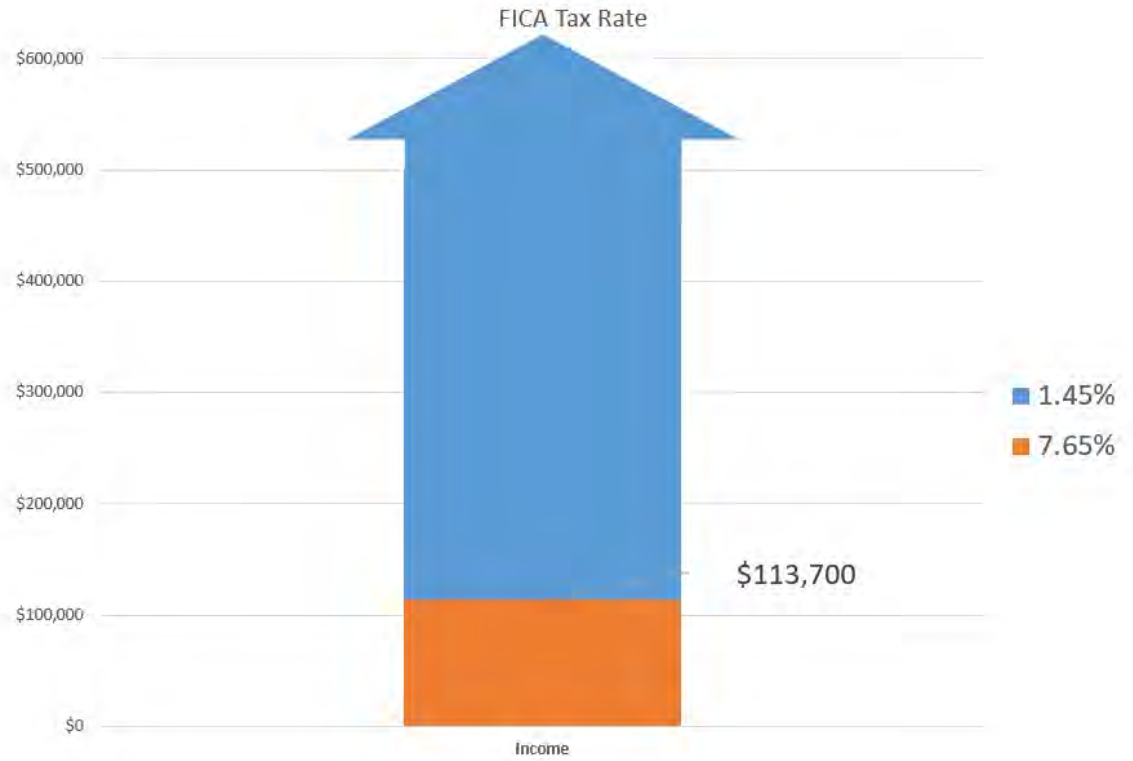
$$A = P \left( 1 + \frac{APR}{n} \right)^{nY}, \quad A = Pe^{APR \times Y}, \quad A = P + APR \times P \times Y$$

$$PMT = \frac{P \times \frac{APR}{n}}{\left[ 1 - \left( 1 + \frac{APR}{n} \right)^{(-nY)} \right]}, \quad A = PMT \times \frac{\left[ \left( 1 + \frac{APR}{n} \right)^{(nY)} - 1 \right]}{\left( \frac{APR}{n} \right)}$$



# Test 2 Note Sheet

---



## 3 Exam 3 Topics

### 3.1 Counting Methods

The methods of counting that we will explore fall into three main approaches: the slot argument,  $nPr$ , and  $nCr$ . These methods are best explored by viewing examples. So, let's see some examples.

#### 3.1.1 Slot Argument

The slot argument is best explained by use of examples.

**Example 3.1.** *A restaurant offers a three course meal which consists of an appetizer, main course, and dessert. If there are 5 appetizers, 3 main courses, and 4 desserts how many three course meals can be ordered?*

*We use an empty slot for each course and fill in each slot with the number of options we have for each course.*

$$\begin{array}{c} \underline{\quad} \times \underline{\quad} \times \underline{\quad} \\ \underline{5} \times \underline{3} \times \underline{4} = 60 \end{array}$$

*Hence there are 60 different three course meals that are possible.*

**Example 3.2.** *Licenses plates in a particular state require three letters followed by four numbers with no repeated letters or numbers allowed. How many license plates can be made?*

*Recall the fact that there are 26 letters of the alphabet (A-Z) and 10 numbers (0-9). Once a letter or number is chosen the number of options drops by 1 due to the fact that no repeats are allowed.*

$$\begin{array}{c} \underbrace{\quad \times \quad \times \quad}_{\text{letters}} \times \underbrace{\quad \times \quad \times \quad \times \quad}_{\text{numbers}} \\ \underline{26} \times \underline{25} \times \underline{24} \times \underline{10} \times \underline{9} \times \underline{8} \times \underline{7} = 78,624,000 \end{array}$$



Hence over 78 million different license plates can be made.

**Example 3.3.** *Six students are seated in six chairs. How many seating arrangements can be made?*

*We will invoke the slot method for solving. See that we have six chairs which we will think of as six empty slots. In each slot we will place the number of options for students that can be seated. Notice, once one person sits down we have one less option for the remaining seats.*

$$\underline{6} \times \underline{5} \times \underline{4} \times \underline{3} \times \underline{2} \times \underline{1} = 720$$

### 3.1.2 Factorial

This stair-stepping down multiplication used in the previous examples is very common within counting methods. Consequently, we will introduce a new operation called the factorial.

$$n! = n \times (n - 1) \times (n - 2) \times \dots \times 2 \times 1 \quad (3.1)$$

Hence we may express  $6 \times 5 \times 4 \times 3 \times 2 \times 1$  as  $6!$  which reads “six factorial”.

**Example 3.4.** *Six students are seated in four chairs. How many seating arrangements can be made?*

*Again, we will utilize the slot method for solving. Notice that a seating arrangements of “Ashley, Bob, Carl, Dennis” is different from “Dennis, Carl, Bob, Ashley”.*

$$\underline{6} \times \underline{5} \times \underline{4} \times \underline{3} = 360$$

We use the previous example as a foundation to derive the counting functions  $nPr$  and  $nCr$ .

### 3.1.3 $nP_r$

From the previous example, we see that there are  $6 \times 5 \times 4 \times 3 \times 2$  ways that six students can be seated in four chairs. We will express this using factorials to derive the counting function  $nPr$ .

$$\begin{aligned}
6 \times 5 \times 4 \times 3 &= \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} \\
&= \frac{6!}{2!} = \frac{6!}{(6-4)!}
\end{aligned}$$

See that the denominator of  $(6-4)!$  is coming from the six students and four chairs. Hence we have obtained a general strategy for tackling a question in which we have  $n$  objects and select  $r$  of them where the order matters (the order that the students were seated mattered).

Thus we have a general equation to answer, “How many ways can  $r$  objects be selected from a pool of  $n$  objects where order matters”.

$${}_n P_r = \frac{n!}{(n-r)!} \tag{3.2}$$

### 3.1.4 ${}_n C_r$

It is natural to ask, “How many seating arrangements, with four chairs and six students, can be made if order doesn’t matter?” We can obtain a solution with a slight adjustment to our previous example by dividing by the number of ways we can arrange 4 objects.

**Example 3.5.** *Six students are seated in four chairs. How many seating arrangements can be made if the order in which the students are seated doesn’t matter?*

*Notice we may recycle our previous solution and divide by the number of ways we can arrange four objects. To determine the number of ways we can arrange four objects think of the slot argument.*

$$\frac{6!}{(6-4)!4!} = \frac{6!}{(6-4)!} \times \frac{1}{4!} = \frac{6!}{(6-4)!4!} = 15$$

*We now have a general strategy to answer the question, “How many ways can  $r$  objects be selected from a pool of  $n$  objects where order does not matter?”*

$${}_n C_r = \frac{n!}{(n-r)!r!} \tag{3.3}$$

We now put our counting functions to use in order to solve the following example.

**Example 3.6.** *A lottery consists of selecting 6 numbers from the numbers 1-13. How many ways can 6 numbers be selected if order does matter? What if order doesn't matter?*

*If order matters,*

$${}_{13}P_6 = 1,235,520$$

*If order doesn't matter,*

$${}_{13}C_6 = 1,716$$

## 3.2 Probability

We now explore the idea of probability. In general, the probability that “event A” will occur is,

$$P(A) = \frac{\text{number of ways event } A \text{ can occur}}{\text{total possible outcomes}} \quad (3.4)$$

where  $P(A)$  reads “the probability of A”. We do a quick example in order to find our bearings.

**Example 3.7.** *What is the probability of rolling an even number with a fair six-sided die?*

*Note there are six possible outcomes with a fair six-sided die (1,2,3,4,5,6) where three outcomes are even (2,4,6). Hence we have,*

$$P(\text{even}) = \frac{3}{6} = \frac{1}{2}$$

We may also find the probability that “event A” does not occur.

$$P(\text{event } A \text{ does NOT occur}) = 1 - P(A) \quad (3.5)$$

**Example 3.8.** *A fair six-sided die is rolled. What is the probability of not rolling an even number?*

$$\begin{aligned}P(\text{not even}) &= 1 - P(\text{even}) \\P(\text{not even}) &= 1 - \frac{3}{6} \\P(\text{not even}) &= \frac{6}{6} - \frac{3}{6} = \frac{6-3}{6} = \frac{3}{6} = \frac{1}{2}\end{aligned}$$

We now do some more involved examples.

**Example 3.9.** *Find the probability of rolling a sum of 8 with two fair six-sided dice.*

$$P(\text{sum of } 8) = \frac{\text{number of ways to roll a sum of } 8}{\text{total possible outcomes}}$$

*Hence we must determine the number of ways to roll a sum of 8 with two fair dice as well as the total possible outcomes when two fair dice are rolled.*

Table 3.1: Ways to Roll a Sum of 8 with two dice

6 and 2  
5 and 3  
4 and 4  
3 and 5  
2 and 6

*Note that if the numbers are different we count it twice and if the numbers are the same we only count it once. When we attach colors to the dice it becomes more clear why we do this. We now use the slot argument to determine the total possible outcomes. Note that we have two die, each has six possible outcomes. Hence the total number of outcomes when two six-sided dice are rolled is  $6 \times 6 = 36$ .*

*Thus,*

$$P(\text{rolling a sum of } 8) = \frac{\text{number of ways to roll a sum of } 8}{\text{total possible outcomes}}$$

$$P(\text{rolling a sum of } 8) = \frac{5}{36}$$

**Example 3.10.** Find the probability of not rolling a sum of 8 with two fair six-sided dice.

From the previous example we know the probability of rolling a sum of 8 is  $\frac{5}{36}$ . Hence the probability of not rolling a sum of 8 is,

$$P(\text{NOT rolling a sum of } 8) = 1 - \frac{5}{36}$$
$$P(\text{NOT rolling a sum of } 8) = \frac{36}{36} - \frac{5}{36} = \frac{36 - 5}{36} = \frac{31}{36}$$

### 3.2.1 AND Probability

We now introduce the idea of multiple events in probability. We first explore AND probability. The probability that two events A AND B both occur is given by the equation listed below.

$$P(A \text{ and } B) = P(A) \times P(B) \tag{3.6}$$

AND probability falls into two categories: independent events and dependent events. We do some quick examples to illustrate AND probability with independent events.

#### 3.2.1.1 Independent Events

**Example 3.11.** A fair coin is flipped twice. What is the probability that the coin lands as tails both times?

$$P(\text{tails and tails}) = P(\text{tails}) \times P(\text{tails})$$

$$P(\text{tails and tails}) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

**Example 3.12.** A fair coin is flipped twelve times. What is the probability that the coin lands as tails all twelve times?

$$P(12 \text{ tails}) = \underbrace{P(\text{tails}) \times P(\text{tails}) \times \dots \times P(\text{tails})}_{12 \text{ times}}$$

$$P(12 \text{ tails}) = \underbrace{\frac{1}{2} \times \frac{1}{2} \times \dots \times \frac{1}{2}}_{12 \text{ times}} = \frac{1}{2^{12}} = \frac{1}{4096}$$

**Example 3.13.** Find the probability of rolling a 1 and then a 5 and then a 4 with a fair six-sided die.

$$P(1 \text{ and } 5 \text{ and } 4) = \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{216}$$

**Example 3.14.** Find the probability of rolling three sixes in a row with a fair six-sided die.

$$P(6 \text{ and } 6 \text{ and } 6) = \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{216}$$

AND probability comes in two styles: independent events and dependent events. The above examples demonstrated independent events because the first coin flip did not effect the second coin flip and so forth (similarly the first roll of the die did not effect the outcome of the second roll). This becomes obvious upon recognition that the probability of getting a tails is always 1/2 and the probability of getting any given number on a die is 1/6. Thus the above events were independent.

We now do an example of dependent events.

### 3.2.1.2 Dependent Events

**Example 3.15.** Find the probability of drawing four face cards in a row, from a standard deck of cards, without returning the drawn cards to the deck.

See that each time we draw a card from the deck and do not return it we change the deck. Thus the act of not returning the card makes these events dependent. Again, notice the total possible outcomes drops by one because the deck contains one less card than it did before. Note there are 12 face cards in a standard deck (assuming ace is not a face) at the start. Hence the probability of the combined events is given by,

$$P(\text{four face cards}) = \frac{12}{52} \times \frac{11}{51} \times \frac{10}{50} \times \frac{9}{49}$$

To present the solution as a simplified fraction refer back to the topics from exam one. This stair-stepping multiplication is a red flag for dependent events.

### 3.2.2 OR Probability

When dealing with OR style probability we need to consider if the events are overlapping or not. In general we may find the probability of accomplishing event A or event B by the equation,

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) \quad (3.7)$$

where  $P(A \text{ and } B)$  denotes the overlap between events  $A$  and  $B$ .

We now do an example in order to illustrate overlapping events in order to clarify what a “overlap” is.

#### 3.2.2.1 Overlapping Events

**Example 3.16.** *What is the probability of drawing club or a queen from a standard deck of cards?*

*Note there are 52 cards in a standard deck. The standard deck of cards contains 13 clubs and 4 queens. Because there is a club that is also a queen these events are overlapping. The overlap is the event in which both events can be accomplished simultaneously.*

*Hence we have,*

$$P(\text{club or queen}) = P(\text{club}) + P(\text{queen}) - P(\text{club and queen})$$

$$P(\text{club or queen}) = \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

*Notice that the subtraction of the overlap is used because we double counted the queen of clubs (it was counted when we counted the number of clubs and the number of queens).*

Figure 3.1 illustrates these two overlapping events to further clarify why these events are considered overlapping.

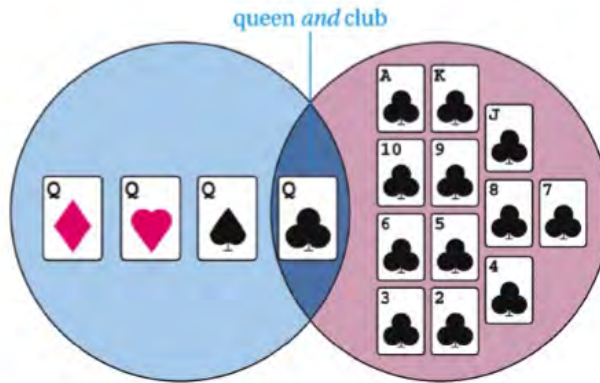


Figure 3.1: Overlapping events of drawing a Queen or a Club

### 3.2.2.2 Non-Overlapping Events

We now do an example to explore non-overlapping events.

**Example 3.17.** Find the probability of rolling either a sum of 7, 8, or 12 with two fair dice.

Note these events are non-overlapping because we cannot roll any two of the three sums at once. Hence there is no overlap. So we have,

$$P(\text{sum of 7, 8, or 12}) = P(\text{sum of 7}) + P(\text{sum of 8}) + P(\text{sum of 12})$$

We follow the strategy introduced earlier to determine the number of ways we can roll a sum of 7, 8, and 12.

$$P(\text{sum of 7, 8, or 12}) = \frac{6}{36} + \frac{5}{36} + \frac{1}{36}$$

$$P(\text{sum of 7, 8, or 12}) = \frac{6 + 5 + 1}{36} = \frac{12}{36} = \frac{1}{3}$$

### 3.2.3 Expected Outcome

We finish our exploration of probability by giving it traction in reality. We do this with the idea of expected outcomes.



**Example 3.18.** *An insurance company has the following data:*

- *1 in 50 policyholders will file a claim for \$2,000.*
- *1 in 20 policyholders will file a claim for \$1,000.*
- *1 in 10 policyholders will file a claim for \$500.*

*If the insurance company charges \$200 for each policy, what is the expected value of the policy? If the company sells 1,000 policies what is the expected profit?*

$$\begin{aligned} \text{Expected Value} &= \text{Price} - \text{Liability} \\ \text{Expected Value} &= 200 - \underbrace{\left( \frac{1}{50} \times 2000 + \frac{1}{20} \times 1000 + \frac{1}{10} \times 500 \right)}_{\text{Liability}} \\ \text{Expected Value} &= \$60 \end{aligned}$$

*Where the expected profit is,*

$$\begin{aligned} \text{Expected Profit} &= \text{Expected Value} \times \text{Number Sold} \\ \text{Expected Profit} &= 60 \times 1000 = 60,000 \end{aligned}$$

### 3.3 Introduction to Statistics

Statistics has two meanings. The first, the science of data collecting. The second, a collection of data values.

Within statistics we want to know something about a population. We do this by studying a sample of the population. Note that a sample is a subset of the given population. The population parameters refer to what one wants to know about the population where the sample statistics refers to the information that was obtained by studying the sample.

We wish to make a statistical inference onto the population by studying a sample. That is to say, use information obtained from the sample in order to obtain information about the population. This is why a representative sample is important. A

representative sample is a sample that reflects the broader population. We will address the requirements to make statistical inferences more completely later when we study normal distributions. For now, we accept the fact that we need a representative sample in order to make statistical inferences from a sample onto a population.

We introduce some fundamental concepts in statistics.

### Sampling Methods

- Simple random sampling requires each item chosen to have an equal likelihood of being selected.
- Systematic sampling uses a simple system to choose the sample, such as every 10th item.
- Convenience sampling constructs samples out of convenience, such as all people in the same classroom.
- Stratified sampling is used when comparing subgroups of a given population.

### Five Steps in a Statistical Study

- (1) Determine the population you want to study and what exactly you want to learn about it.
- (2) Choose a representative sample from the population.
- (3) Collect raw data from the sample and summarize this data by finding sample statistics of interest.
- (4) Use the sample statistics to infer the population parameters.
- (5) Draw conclusions. If a statistical study tends to favor a particular result, we say it is biased.

### 8 Rules to Read Statistics

- (1) Get a big picture view of the study.

- (2) Consider the source.
- (3) Look for bias in the sample.
- (4) Look for problems in defining or measuring the variables of interest.
- (5) Be aware of confounding variables.
- (6) Consider the setting and wording in the surveys.
- (7) Check that the results are presented fairly.
- (8) Stand back and consider the conclusion.

### Types of Statistical Studies

- **Observational Study:** In an observational study researchers observe characteristics of the sample but do not try to influence or modify these characteristics.
- **Experimental Study:** In an experiment researchers apply treatment to the sample and observe its effects.

### Types of Experiments

- **Blind:** Participants do not know which participants are getting the treatment.
- **Double Blind:** Participants and researchers don't know who is getting the treatment.

### Type of Observational Study

- **Case Control Study:** Usually for ethical reasons the control group and treatment group split themselves up based on their individual behaviors.

Note that a control group is the group that does not get treatment and the treatment group is the group that does get the treatment. This is done in order to study the effects of the treatment.

## 3.4 Fundamental Statistical Measurements

We wish to find some stated information about a given population. This by its very nature causes a problem. How can one determine information about a population by only studying a sample which is a subset of the population? This issue is resolved (to some extent) by recognition of normal distributions and where they come from. When the time comes we will investigate normal distributions. For now we study fundamental statistical measurements in order to prepare ourselves for the study of normal distributions.

### 3.4.1 Mean, Median, and Mode

The following definitions are provided as a brief review.

- Mean: The sum of data values divided by the number of data values.
- Median: The data value that lives in the middle of the ordered data set.
- Mode: The most occurring data value in the data set.

A quick example should help shake the rust off.

**Example 3.19.** Find the mean, median, and mode of the following set:  $\{1, 3, 2\}$

The mean is  $\frac{1 + 2 + 3}{3} = 2$ .

The median is 2.

There is no mode. Recall the mode is the most occurring value. Recognize that all of the data values occur exactly once and hence there is no most occurring value.

**Example 3.20.** We will find the mean median and mode of the following set:  $\{2, 2, 3, 3, 4, 4, 4, 4\}$ .

The mean is  $\frac{26}{8} = 3.25$ .

The median is  $\frac{3 + 4}{2} = 3.5$  because we have an even number of data values we find the number that lives in the middle of 3 and 4.

The mode is 4.

### 3.4.2 5 Number Summary

A five-number summary consists of five numbers: the low value, lower quartile value, middle value, upper quartile value, and the high value. The lower quartile (also called Q1) is the middle value of the lower half where the upper quartile (also called Q3) is the middle value of the upper half.

Let's see an example.

**Example 3.21.** Find the five-number summary of the data set  $\{1, 2, 3, 4, 5, 6, 7\}$ .

See that the low, middle, and high values are 1, 4, and 7 respectively. To find the lower quartile we dissect the set into the lower and upper halves.

$$\begin{array}{ccc} & \text{median} & \\ & \frown & \\ 1, 2, 3, & 4, & 5, 6, 7 \\ \underbrace{\hspace{1.5cm}} & & \underbrace{\hspace{1.5cm}} \\ \text{lower half} & & \text{upper half} \end{array}$$

Thus the lower quartile is 2 and the upper quartile is 6.

**Example 3.22.** Find the five-number summary of the data set  $\{1, 2, 3, 4, 5, 6\}$ .

See that the low, middle, and high values are 1, 3.5, and 6 respectively. To find the lower quartile we dissect the set into the lower and upper halves.

$$\begin{array}{ccc} & \text{upper half} & \\ & \frown & \\ 1, 2, 3, & 4, 5, 6 & \\ \underbrace{\hspace{1.5cm}} & & \\ \text{lower half} & & \end{array}$$

Thus the lower quartile is 2 and the upper quartile is 5.

#### 3.4.2.1 Box and Whisker Plot

A box and whisker plot is a visual representation of the five-number summary. Figure 3.2 illustrates a five-number summary shown with a box and whisker plot.

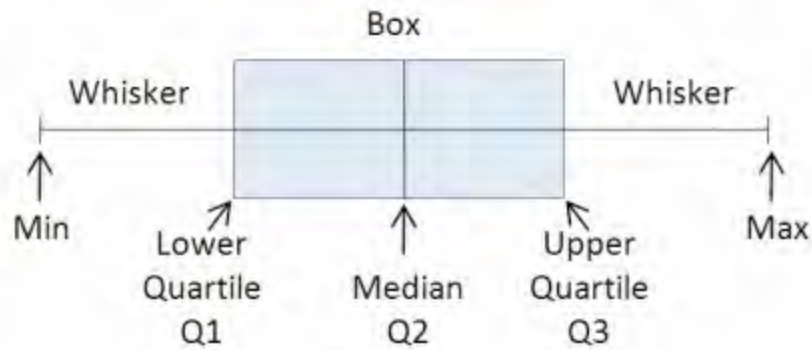


Figure 3.2: Box and whisker plot

### 3.4.3 Standard Deviation ( $\sigma$ )

The standard deviation measures how narrow or widely spread the data values are about the mean. The standard deviation is a positive number that is typically denoted with the Greek letter sigma ( $\sigma$ ). The larger the standard deviation the more widely spread the data values are about the mean. Figure 3.3 shows the correlation between the standard deviation and the spread of data values.

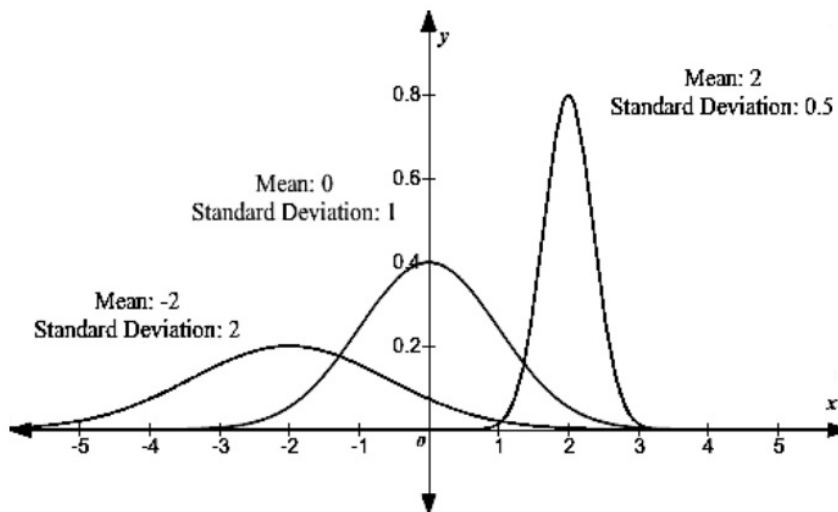


Figure 3.3: The larger the standard deviation the more widely spread the data values are about the mean

The standard deviation is computed using the steps given below.

### Steps to Find the Standard Deviation

- (1) Compute the mean of the data set.
- (2) For each data value find its deviation from the mean by:  $\text{Data Value} - \text{Mean} = \text{Deviation}$
- (3) Square each deviation found above.
- (4) Sum all of the squared deviations.
- (5) Divide the above sum of squared deviations by the number of data values minus one.
- (6) Take the square root of the above value in order to obtain the standard deviation.

## 3.5 Distributions of Data

We now introduce the different types of data distributions that we may encounter. A picture says a thousand words so we introduce figures to help conceptualize distributions of data.

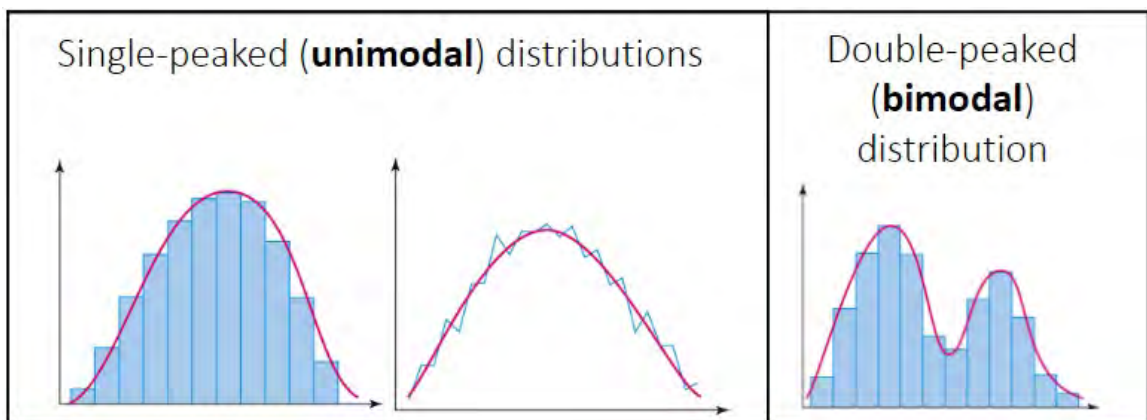


Figure 3.4: Modal and bi-modal distributions

### 3.5.1 Skewed Distributions

Skewed distributions occur when outliers are present within a distribution. An outlier is a data value that does not live within the main group of data values. A left skewed distribution has outliers at the low end of the distribution which causes the mean to be pulled to the left of the median. Similarly, a right skewed distribution has outliers at the high end of the distribution which causes the mean to be pulled to the right of the median.

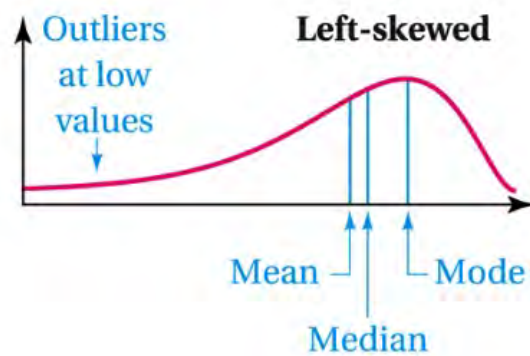


Figure 3.5: Left skewed distribution

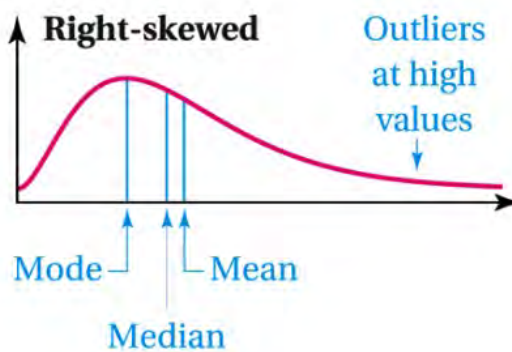


Figure 3.6: Right skewed distribution



### 3.5.2 Symmetric Distributions

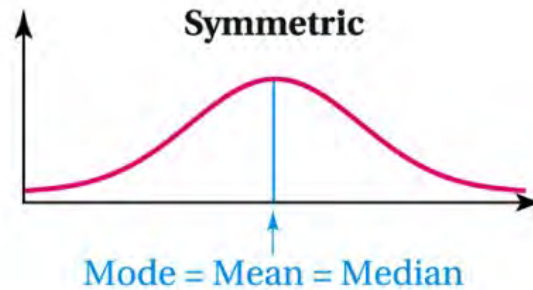


Figure 3.7: Symmetric distribution

## 3.6 Normal Distributions

A normal distribution is a special type of symmetric distribution (mean = median = mode). A normal distribution is characterized by how the data falls about the mean. The data in a normal distribution will follow the 68,95,99.7 rule. This rule is illustrated below.

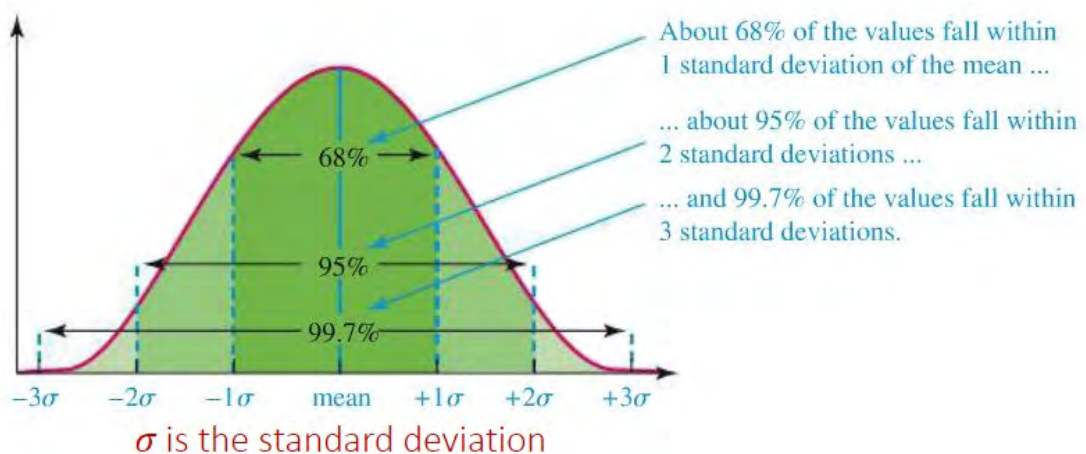


Figure 3.8: Normal distribution with the 68, 95, 99.7 rule illustrated

### 3.6.1 68, 95, 99.7 Rule

We now use the 68, 95, 99.7 rule in the following example in order to get a better understanding of normal distributions.

**Example 3.23.** *You collected 1,000 data values and found a mean of 55 and a standard deviation of 10. If the data is normally distributed, then it will follow the 68,95,99.7 rule.*

*It follows that 68% of the data values will be within one standard deviation of the mean. More explicitly,  $\underbrace{680}_{.68 \times 1,000}$  data values will be between  $\underbrace{45}_{55 - 10}$  and  $\underbrace{65}_{55 + 10}$ .*

*It follows that 95% of the data values will be within two standard deviation of the mean. More explicitly,  $\underbrace{950}_{.95 \times 1,000}$  data values will be between  $\underbrace{35}_{55 - 2 \times 10}$  and  $\underbrace{75}_{55 + 2 \times 10}$ .*

*It follows that 99.7% of the data values will be within three standard deviation of the mean. More explicitly,  $\underbrace{997}_{.997 \times 1,000}$  data values will be between  $\underbrace{25}_{55 - 3 \times 10}$  and  $\underbrace{85}_{55 + 3 \times 10}$ .*

### 3.6.2 Using Z-chart to Find Percentiles

A percentile informs the reader how a particular data value compares to all other data values. For instance, if a data value is in the 50th percentile then it is bigger than 50 percent of all other data values. Similarly, if a data value is in the 25th percentile then it is larger than 25 percent of all other data values.

A z-chart connects a data value to its percentile by use of a z-score. A z-score is the number of standard deviations the data value is above or below average. For example, if a data value is one standard deviation above average is has a z-score of 1. Similarly, if a data value is 2 standard deviations below average then it has a z-score of -2. Thus, data values that are below average have a negative z-score and data values that are above average have a positive z-score.

Given a data value, mean ( $\mu$ ), and standard deviation ( $\sigma$ ), the z-score is computed with Equation 3.8.

$$z = \frac{\text{Data Value} - \mu}{\sigma} \quad (3.8)$$

Given a percentile, mean ( $\mu$ ), and standard deviation ( $\sigma$ ), the data value is computed with Equation 3.9.

$$Data\ Value = \mu + z\sigma \tag{3.9}$$

We use the following flow chart to navigate from a data value to its percentile.

$$Data\ Value \xrightarrow{Equation\ 3.8} z - score \xrightarrow{z-chart} Percentile$$

We may apply this process in the opposite direction. That is to say, given a percentile we may obtain the data value by use of the following flow chart.

$$Percentile \xrightarrow{z-chart} z - score \xrightarrow{Equation\ 3.9} Data\ Value$$

### 3.6.3 Central Limit Theorem

Recall that we wish to find information about a given population by studying a sample. This by its very nature causes a problem. How can one determine information about a population by only studying a sample which is a subset of the population? Some quick answers would be, “A representative sample does the trick.” However, how does one know that we have a representative sample? Or, “A randomly selected sample is sure to be a representative sample.” But how does one quantify randomness? Again, this issue is resolved (to some extent) by recognition of the Central Limit Theorem.

The Central Limit Theorem states that we may make a statistical inference from a sample onto a greater population if the sample is of sufficient size and randomly selected. We accept that the sample is of sufficient size and randomly selected when a normal distribution is found within the sample statistics. We know when a normal distribution is present by the use of the 68,95,99.7% rule.

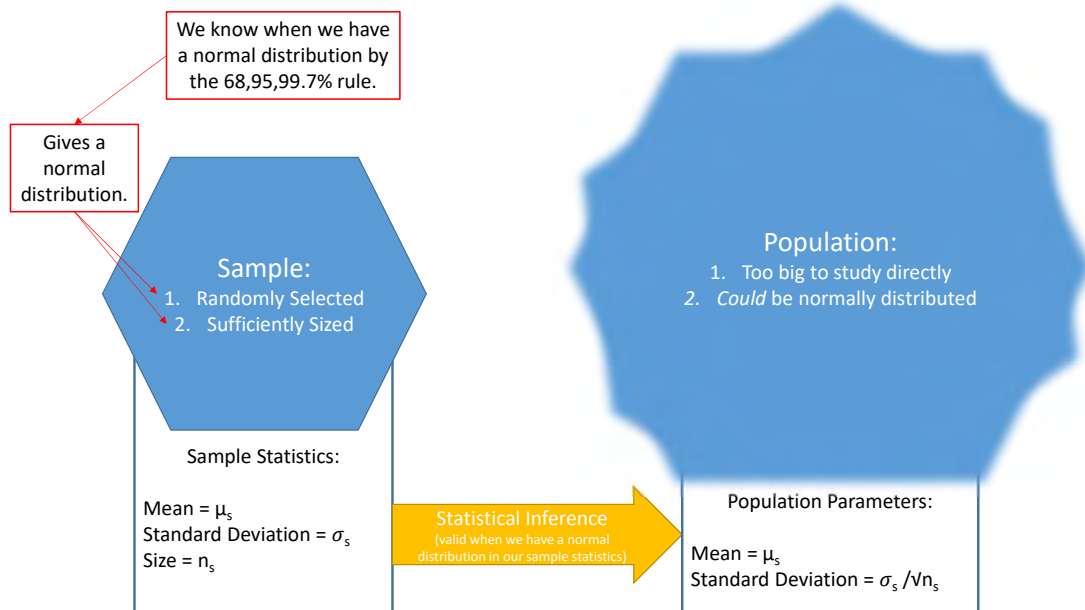


Figure 3.9: Illustration of the Central Limit Theorem

## 3.7 Test 3 Note Sheet

There is no note sheet for the third exam.  
Microsoft Excel and a calculator are allowed/encouraged.

## 4 Topics after Exam 3

### 4.1 Linear Modeling

**Example 4.1.** *Posters*

**Example 4.2.** *raffle tickets*

**Example 4.3.** *washer depreciation*

### 4.2 Exponential Modeling

Before we begin the discussion of exponential modeling we refresh ourselves with where/why we need logarithms. Please do not be scared for logarithms are much more simple than your last math teacher made them out to be.

#### 4.2.1 Logarithms

#### 4.2.2 Exponential Growth

When a starting quantity,  $Q_0$ , grows exponentially at a rate of  $r$  during a specified period of time,  $t$ , we may determine the current (or future) quantity,  $Q$ , by using the equation listed below. Note that  $r$  is typically stated as a percent but the decimal notation is used when computing (exactly like APR from the finance topics).

$$Q = Q_0(1 + r)^t \tag{4.1}$$

Depending upon the information that is provided, we may need to rephrase Equation 4.1. We use this rephrased equation if we do not know the original quantity and only know the portion (a fraction or percent) of the original quantity that is desired.

$$\frac{Q}{Q_0} = (1 + r)^t \quad (4.2)$$

**Example 4.4.** *A population grows by 5% each year. When will the population double?*

**Example 4.5.** *A population of 1,000 grows by 6% each year. When will the population reach 2,200?*

### 4.2.3 Exponential Decay

When a starting quantity,  $Q_0$ , decreases exponentially at a rate of  $r$  during a specified period of time,  $t$ , we may determine the current (or future) quantity,  $Q$ , by using the equation listed below. Note, again, that  $r$  is typically stated as a percent but the decimal notation is used when computing (again, exactly like APR from the finance topics).

$$Q = Q_0(1 - r)^t \quad (4.3)$$

If we are not provided the original amount  $Q_0$  and only provided a portion (fraction or percent) of the quantity that remains we rephrase Equation 4.3 into the equation given below.

$$\frac{Q}{Q_0} = (1 - r)^t \quad (4.4)$$

It is common to explore half-life examples when working with exponential modeling. As you might anticipate half-life is a special case of exponential decay. Note that, within the vein of half-life, the starting quantity  $Q_0$  decreases by 50% over a half-life. Hence, the half-life follows Equation 4.3 with a slight adjustment to the exponent where  $t_H$  is the half-life of the given element.

$$Q = Q_0(0.5)^{\frac{t}{t_H}} \quad (4.5)$$

Just as before, we may need to rewrite Equation 4.5 in order to accommodate situations where we do not know the starting quantity. We do this and obtain the equation given below.

$$\frac{Q}{Q_0} = (0.5)^{\frac{t}{t_H}} \quad (4.6)$$

**Example 4.6.** *Decay by 5% each year. Find the population after 14 years. When will the population be 60?*

**Example 4.7.** *Half life of 3.5 billion years. 60 percent remains. How old is the rock.*